INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

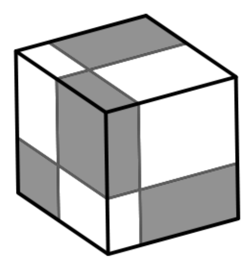
SENIOR PAPER: YEARS 11, 12

Tournament 43, Northern Fall 2021 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. We call a positive integer k *interesting* if the product of the first k primes is divisible by k. For example the product of the first two primes is $2 \cdot 3 = 6$, it is divisible by 2, hence 2 is an interesting integer. What is the maximum possible number of consecutive interesting integers? (3 points)

2. A cube is split into 8 boxes by three planes parallel to its faces. The resulting parts are painted in a chessboard pattern. The volumes of the black boxes are 1, 6, 8, 12. Find the volumes of the white boxes. (4 points)



3. In a square grid of size 2021×2021 all the cells are initially white. Ivan selects two cells and paints them black. After that, all the other cells are iteratively painted black. At each step, all the cells that share a side with at least one black cell are simultaneously painted black. Ivan selects the two initial cells so that the entire square is painted black as fast as possible. How many steps will this take? (6 points)

4. A segment AB is given in three-dimensional space. Three points X, Y, Z are picked in the space so that ABX is an equilateral triangle and $ABYZ$ is a square. Prove that the orthocenters of all triangles XYZ obtained in this way belong to a fixed circle. (6 points)

5. Initially, we are given the segment $[0, 1]$. At each step we may split one of the available segments into two new segments and write the product of the lengths of these two new segments onto a blackboard. Prove that the sum of the numbers on the blackboard will never exceed $1/2$. (6 points)